Contraction After Small Transients

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Abstract

Contraction theory is a powerful tool for proving asymptotic properties of nonlinear dynamical systems including convergence to an attractor and entrainment to a periodic excitation. We consider three generalizations of contraction with respect to a norm that allow contraction to take place after small transients in time and/or amplitude. These generalized contractive systems (GCSs) are useful for several reasons. First, we show that there exist simple and checkable conditions guaranteeing that a system is a GCS, and demonstrate their usefulness using several models from systems biology. Second, allowing small transients does not destroy the important asymptotic properties of contractive systems like convergence to a unique equilibrium point, if it exists, and entrainment to a periodic excitation. Third, in some cases as we change the parameters in a contractive system it becomes a GCS just before it looses contractivity with respect to a norm. In this respect, generalized contractivity is the analogue of marginal stability in Lyapunov stability theory.

Index Terms

Differential analysis, contraction, stability, entrainment, phase locking, systems biology.

I. Introduction

Differential analysis is based on studying the time evolution of the distance between trajectories emanating from different initial conditions. A dynamical system is called contractive if any two trajectories converge to one other at an exponential rate. This implies many desirable properties including convergence to a unique attractor (if it exists), and entrainment to periodic excitations [2], [3], [4]. Contraction theory proved to be a powerful tool for analyzing nonlinear dynamical systems, with applications in control theory [5], observer design [6], synchronization of coupled oscillators [7], and more. Recent extensions include: the notion of partial contraction [8], analyzing networks of interacting agents using contraction theory [9], [10], a Lyapunov-like characterization of incremental stability [11], and a LaSalle-type principle for contractive systems [12]. There is also a growing interest in design techniques providing controllers that render control systems contractive or incrementally stable; see, e.g. [13] and the references therein, and also the incremental ISS condition in [14]).

A contractive system with added diffusion terms or random noise still satisfies certain asymptotic properties [15], [16]. In this respect, contraction is a robust property.

In this paper, we introduce three forms of generalized contractive systems (GCSs). These are motivated by requiring contraction with respect to a norm to take place only after arbitrarily small transients in time and/or amplitude. Indeed, contraction is usually used to prove *asymptotic* properties, and thus allowing (arbitrarily small) transients seems reasonable. In some cases as we change the parameters in a contractive system it becomes a GCS just before it looses contractivity. In this respect, a GCS is the analogue of marginal stability in Lyapunov stability theory. We provide several sufficient conditions for a system to be a GCS. These conditions are checkable, and we demonstrate their usefulness using examples of systems that are *not* contractive with respect to any norm, yet are GCSs.

We begin with a brief review of some ideas from contraction theory. For more details, including the historic development of contraction theory, and the relation to other notions, see e.g. [17], [18], [19].

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Consider the time-varying system

$$\dot{x} = f(t, x), \tag{1}$$

with the state x evolving on a positively invariant convex set $\Omega \subseteq \mathbb{R}^n$. We assume that $f(t, x)$ is differentiable with respect to x, and that both $f(t, x)$ and $J(t, x) := \frac{\partial f}{\partial x}(t, x)$ are continuous in (t, x). Let $x(t, t_0, x_0)$ denote the solution of (1) at time $t \geq t_0$ with $x(t_0) = x_0$ (for the sake of simplicity, we assume from here on that $x(t, t_0, x_0)$ exists and is unique for all $t \geq t_0 \geq 0$ and all $x_0 \in \Omega$).

We say that (1) is *contractive* on Ω with respect to a norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ if there exists $c > 0$ such that

$$|x(t_2, t_1, a) - x(t_2, t_1, b)| \leq \exp(-(t_2 - t_1)c)|a - b| \tag{2}$$

for all $t_2 \geq t_1 \geq 0$ and all $a, b \in \Omega$. In other words, any two trajectories contract to one another at an exponential rate. This implies in particular that the initial condition is "quickly forgotten". Note that Ref. [2] provides a more general and intrinsic definition, where contraction is with respect to a time- and state-dependent metric $M(t, x)$ (see also [20] for a general treatment of contraction on a Riemannian manifold). Some of the results below may be stated using this more general framework. But, for a given dynamical system finding such a metric may be difficult. Another extension of contraction is incremental stability [11]. Our approach is based on the fact that there exists a simple sufficient condition guaranteeing (2), so generalizing (2) appropriately leads to *checkable* sufficient conditions for a system to be a GCS. Another advantage of our approach is that a GCS retains the important property of entrainment to periodic signals.

Recall that a vector norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ induces a matrix measure $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$ defined by $\mu(A) := \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon}(||I + \epsilon A|| - 1)$, where $||\cdot|| : \mathbb{R}^{n\times n} \to \mathbb{R}_+$ is the matrix norm induced by $|\cdot|$. A standard approach for proving (2) is based on bounding some matrix measure of the Jacobian J. Indeed, it is well-known (see, e.g. [3]) that if there exist a vector norm $|\cdot|$ and $c > 0$ such that the induced matrix measure $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$ satisfies

$$\mu(J(t, x)) \leq -c, \tag{3}$$

for all $t_2 \geq t_1 \geq 0$ and all $x \in \Omega$ then (2) holds. (This is in fact a particular case of using a Lyapunov-Finsler function to prove contraction [12].)

It is well-known [21, Ch. 3] that the matrix measure induced by the L_1 vector norm is

$$\mu_1(A) = \max\{c_1(A), \ldots, c_n(A)\}, \tag{4}$$

where

$$c_j(A) := A_{jj} + \sum_{\substack{1 \leq i \leq n \\ i \neq j}} |A_{ij}|, \tag{5}$$

i.e., the sum of the entries in column j of A, with non diagonal elements replaced by their absolute values. The matrix measure induced by the L_∞ norm is

$$\mu_\infty(A) = \max\{d_1(A), \ldots, d_n(A)\}, \tag{6}$$

where

$$d_j(A) := A_{jj} + \sum_{\substack{1 \leq i \leq n \\ i \neq j}} |A_{ji}|, \tag{7}$$

i.e., the sum of the entries in row j of A, with non diagonal elements replaced by their absolute values.

Often it is useful to work with scaled norms. Let $|\cdot|_*$ be some vector norm, and let $\mu_* : \mathbb{R}^{n\times n} \to \mathbb{R}$ denote its induced matrix measure. If $P \in \mathbb{R}^{n\times n}$ is an invertible matrix, and $|\cdot|_{*,P} : \mathbb{R}^n \to \mathbb{R}_+$ is the vector norm defined by $|z|_{*,P} := |Pz|_*$ then the induced matrix measure is $\mu_{*,P}(A) = \mu_*(PAP^{-1})$.

One important implication of contraction is *entrainment* to a periodic excitation. Recall that $f : \mathbb{R}_+ \times$

$\Omega \to \mathbb{R}^n$ is called T*-periodic* if

$$f(t,x) = f(t+T,x)$$

for all $t \geq 0$ and all $x \in \Omega$. Note that for the system $\dot{x}(t) = f(u(t), x(t))$, with u an input (or excitation) function, f will be T periodic if u is a T-periodic function. It is well-known [2], [3] that if (1) is contractive and f is T-periodic then for any $t_1 \geq 0$ there exists a unique periodic solution $\alpha : [t_1, \infty) \to \Omega$ of (1), of period T, and every trajectory converges to α. Entrainment is important in various applications ranging from biological systems [22], [3] to the stability of a power grid [23]. Note that for the particular case where f is time-invariant, this implies that if Ω contains an equilibrium point e then it is unique and all trajectories converge to e.

The remainder of this paper is organized as follows. Section II presents three generalizations of (2). Section III details sufficient conditions for their existence, and describes their implications. The proofs of all the results are detailed in Section V.

II. Definitions of contraction after small transients

We begin by defining three generalizations of (2).

Definition 1 *The time-varying system* (1) *is said to be:*

- contractive after a small overshoot and short transient *(SOST) on* Ω *w.r.t. a norm* $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ *if for each* $\varepsilon > 0$ *and each* $\tau > 0$ *there exists* $\ell = \ell(\tau, \varepsilon) > 0$ *such that*
$$|x(t_2+\tau, t_1, a) - x(t_2+\tau, t_1, b)| \leq (1+\varepsilon)\exp(-(t_2-t_1)\ell)|a-b| \tag{8}$$
for all $t_2 \geq t_1 \geq 0$ *and all* $a, b \in \Omega$.
- contractive after a small overshoot *(SO) on* Ω *w.r.t. a norm* $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ *if for each* $\varepsilon > 0$ *there exists* $\ell = \ell(\varepsilon) > 0$ *such that*
$$|x(t_2, t_1, a) - x(t_2, t_1, b)| \leq (1+\varepsilon)\exp(-(t_2-t_1)\ell)|a-b| \tag{9}$$
for all $t_2 \geq t_1 \geq 0$ *and all* $a, b \in \Omega$.
- contractive after a short transient *(ST) on* Ω *w.r.t. a norm* $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ *if for each* $\tau > 0$ *there exists* $\ell = \ell(\tau) > 0$ *such that*
$$|x(t_2+\tau, t_1, a) - x(t_2+\tau, t_1, b)| \leq \exp(-(t_2-t_1)\ell)|a-b| \tag{10}$$
for all $t_2 \geq t_1 \geq 0$ *and all* $a, b \in \Omega$.

The definition of SOST is motivated by requiring contraction at an exponential rate, but only after an (arbitrarily small) time τ, and with an (arbitrarily small) overshoot $(1+\varepsilon)$. However, as we will see below when the convergence rate ℓ may depend on ε a somewhat richer behavior may occur. The definition of SO is similar to that of SOST, yet now the convergence rate ℓ depends only on ε, and there is no time transient τ (i.e., $\tau = 0$). In other words, SO is a uniform (in τ) version of SOST. The third definition, ST, allows the contraction to "kick in" only after a time transient of length τ.

It is clear that every contractive system is SOST, SO, and ST. Thus, all these notions are generalizations of contraction. Also, both SO and ST imply SOST and, as we will see below, under a mild technical condition on (1) SO and SOST are equivalent. Figure 2 on p. 12 summarizes the relations between these GCSs (as well as other notions defined below).

The motivation for these definitions stems from the fact that important applications of contraction are in proving *asymptotic* properties. For example, proving that an equilibrium point is globally attracting or that the state-variables entrain to a periodic excitation. These properties describe what happens as $t \to \infty$, and so it seems natural to generalize contraction in a way that allows initial transients in time and/or amplitude.

The next simple example demonstrates a system that does not satisfy (2), but is a GCS.

Example 1 *Consider the* scalar *time-varying system*

$$\dot{x}(t) = -\alpha(t)x(t), \tag{11}$$

with the state x evolving on $\Omega := [-1, 1]$, and $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a class K function (i.e. α is continuous and strictly increasing, with $\alpha(0) = 0$). It is straightforward to show that this system does not satisfy (2) *w.r.t.* any *norm (note that the Jacobian $J(t) = -\alpha(t)$ satisfies $J(0) = 0$), yet it is ST, with $\ell(\tau) = \alpha(\tau) > 0$, for any given $\tau > 0$.*

The next section presents our main results. The proofs are placed in Section V.

III. Main Results

The next three subsections study the three forms of GCSs defined above.

A. *contractive after a small overshoot and short transient*

Just like contraction, SOST implies entrainment to a periodic excitation. To show this, assume that the vector field f in (1) is T periodic. Pick $t_0 \geq 0$. Define $m : \Omega \to \Omega$ by $m(a) := x(T + t_0, t_0, a)$. In other words, m maps a to the solution of (1) at time $T + t_0$ for the initial condition $x(t_0) = a$. Then m is continuous and maps the convex and compact set Ω to itself, so by the Brouwer fixed point theorem (see, e.g. [24, Ch. 6]) there exists $\zeta \in \Omega$ such that $m(\zeta) = \zeta$, i.e. $x(T + t_0, t_0, \zeta) = \zeta$. This implies that (1) admits a periodic solution $\gamma : [t_0, \infty) \to \Omega$ with period T. Assuming that the system is also SOST, pick $\tau, \varepsilon > 0$. Then there exists $\ell = \ell(\tau, \varepsilon) > 0$ such that

$$|x(t - t_0 + \tau, t_0, a) - x(t - t_0 + \tau, t_0, \zeta)| \leq (1 + \varepsilon) \exp(-(t - t_0)\ell)|a - \zeta|,$$

for all $a \in \Omega$ and all $t \geq t_0$. Taking $t \to \infty$ implies that every solution converges to γ. In particular, there cannot be two distinct periodic solutions. Thus, we proved the following.

Proposition 1 *Suppose that the time-varying system* (1)*, with state x evolving on a compact and convex state-space $\Omega \subset \mathbb{R}^n$, is SOST, and that the vector field f is T-periodic. Then for any $t_0 \geq 0$ it admits a unique periodic solution $\gamma : [t_0, \infty) \to \Omega$ with period T, and $x(t, t_0, a)$ converges to γ for any $a \in \Omega$.*

Since both SO and ST imply SOST, Proposition 1 holds for all three forms of GCSs.

Our next goal is to derive a sufficient condition for SOST. One may naturally expect that if (1) is contractive w.r.t. a set of norms $|\cdot|_\zeta$, with, say $\zeta \in (0, p]$, $p > 0$, and that $\lim_{\zeta \to 0} |\cdot|_\zeta = |\cdot|$ then (1) is a GCS w.r.t. the norm $|\cdot|$. In fact, this can be further generalized by requiring (1) to be contractive w.r.t. $|\cdot|_\zeta$ only on suitable subset Ω_ζ of the state-space. This leads to the following definition.

Definition 2 *System* (1) *is said to be* nested contractive *(NC) on Ω with respect to a norm $|\cdot|$ if there exist convex sets $\Omega_\zeta \subseteq \Omega$, and norms $|\cdot|_\zeta : \mathbb{R}^n \to \mathbb{R}_+$, where $\zeta \in (0, 1/2]$, such that the following conditions hold.*

(a) $\cup_{\zeta \in (0,1/2]} \Omega_\zeta = \Omega$*, and*

$$\Omega_{\zeta_1} \subseteq \Omega_{\zeta_2}, \quad \textit{for all } \zeta_1 \geq \zeta_2. \tag{12}$$

(b) For every $\tau > 0$ there exists $\zeta = \zeta(\tau) \in (0, 1/2]$, with $\zeta(\tau) \to 0$ as $\tau \to 0$, such that for every $a \in \Omega$ and every $t_1 \geq 0$

$$x(t, t_1, a) \in \Omega_\zeta, \quad \textit{for all } t \geq t_1 + \tau, \tag{13}$$

and (1) *is contractive on Ω_ζ with respect to $|\cdot|_\zeta$.*

(c) The norms $|\cdot|_\zeta$ converge to $|\cdot|$ as $\zeta \to 0$, i.e., for every $\zeta > 0$ there exists $s = s(\zeta) > 0$, with $s(\zeta) \to 0$ as $\zeta \to 0$, such that

$$(1 - s)|y| \leq |y|_\zeta \leq (1 + s)|y|, \quad \textit{for all } y \in \Omega.$$

Eq. (13) means that after an arbitrarily short time every trajectory enters and remains in a subset Ω_ζ of the state space on which we have contraction with respect to $|\cdot|_\zeta$. We can now state the main result in this subsection.

Theorem 1 *If the system* (1) *is NC w.r.t. the norm* $|\cdot|$ *then it is SOST w.r.t. the norm* $|\cdot|$.

The next example demonstrates Theorem 1. It also shows that as we change the parameters in a contractive system, it may become a GCS when it hits the "verge" of contraction (as defined in (2)). This is reminiscent of an asymptotically stable system that becomes marginally stable as it looses stability.

Example 2 *Consider the system*

$$\begin{aligned}
\dot{x}_1 &= g(x_n) - \alpha_1 x_1, \\
\dot{x}_2 &= x_1 - \alpha_2 x_2, \\
\dot{x}_3 &= x_2 - \alpha_3 x_3, \\
&\vdots \\
\dot{x}_n &= x_{n-1} - \alpha_n x_n,
\end{aligned} \tag{14}$$

where $\alpha_i > 0$, *and*

$$g(u) := \frac{1+u}{k+u}, \quad \text{with } k > 1.$$

As explained in [25, Ch. 4] this may model a simple biochemical feedback control circuit for protein synthesis in the cell. The x_i*s represent concentrations of various macro-molecules in the cell and therefore must be non-negative. It is straightforward to verify that* $x(0) \in \mathbb{R}^n_+$ *implies that* $x(t) \in \mathbb{R}^n_+$ *for all* $t \geq 0$.

Let $\alpha := \prod_{i=1}^n \alpha_i$, *and for* $\varepsilon > 0$ *let*

$$D_\varepsilon := \operatorname{diag}\left(1, \alpha_1 - \varepsilon, (\alpha_1 - \varepsilon)(\alpha_2 - \varepsilon), \dots, \prod_{i=1}^{n-1} (\alpha_i - \varepsilon)\right).$$

We show in Section V that if

$$k - 1 < \alpha k^2 \tag{15}$$

then (14) *is contractive on* $\mathbb{R}^n_+$ *w.r.t. the scaled norm* $|\cdot|_{1,D_\varepsilon}$ *for all* $\varepsilon > 0$ *sufficiently small. If* $k - 1 = \alpha k^2$ *then* (14) *does not satisfy* (2), *w.r.t. any norm, on* $\mathbb{R}^n_+$, *yet it is SOST on* $\mathbb{R}^n_+$ *w.r.t. the norm* $|\cdot|_{1,D_0}$.

Note that for all $x \in \mathbb{R}^n_+$,

$$g'(x_n) = \frac{k-1}{(k+x_n)^2} \leq \frac{k-1}{k^2} = g'(0). \tag{16}$$

Thus (15) *implies that the system satisfies* (2) *if and only if the "total dissipation"* α *is strictly larger than* $g'(0)$.

Using the fact that $g(u) < 1$ *for all* $u \geq 0$ *it is straightforward to show that the set*

$$\Omega_r := r([0, \alpha_1^{-1}] \times [0, (\alpha_1 \alpha_2)^{-1}] \times \dots \times [0, \alpha^{-1}])$$

is an invariant set of the dynamics for all $r \geq 1$. *Thus,* (14), *with* $k - 1 \leq \alpha k^2$, *admits a unique equilibrium point* $e \in \Omega_1$ *and*

$$\lim_{t \to \infty} x(t, a) = e, \quad \text{for all } a \in \mathbb{R}^n_+.$$

This property also follows from a more general result [25, Prop. 4.2.1] that is proved using the theory of irreducible cooperative dynamical systems. Yet the contraction approach leads to new insights. For example, it implies that the distance between trajectories can only decrease, and can also be used to prove entrainment to suitable generalizations of (14) *that include periodically-varying inputs.*

Cells often respond to external stimulus by modification of proteins. One mechanism for this is *phosphorelay* (also called phosphotransfer) in which a phosphate group is transferred through a serial chain of proteins from an initial histidine kinase (HK) down to a final response regulator (RR). The next example uses Theorem 1 to analyze a model for phosphorelay from [26].

Example 3 *Consider the system*

$$\begin{aligned}
\dot{x}_1 &= (p_1 - x_1)c - \eta_1 x_1 (p_2 - x_2), \\
\dot{x}_2 &= \eta_1 x_1 (p_2 - x_2) - \eta_2 x_2 (p_3 - x_3), \\
&\vdots \\
\dot{x}_{n-1} &= \eta_{n-2} x_{n-2} (p_{n-1} - x_{n-1}) - \eta_{n-1} x_{n-1} (p_n - x_n), \\
\dot{x}_n &= \eta_{n-1} x_{n-1} (p_n - x_n) - \eta_n x_n,
\end{aligned} \tag{17}$$

where $\eta_i, p_i > 0$, and $c : [t_1, \infty) \to \mathbb{R}_+$. In the context of phosphorelay [26], $c(t)$ is the strength at time t of the stimulus activating the HK, $x_i(t)$ is the concentration of the phosphorylated form of the protein at the ith layer at time t, and p_i denotes the total protein concentration at that layer. Note that $\eta_n x_n$ is the flow of the phosphate group to an external receptor molecule.

In the particular case where $p_i = 1$ for all i (17) *becomes the* ribosome flow model *(RFM) [27]. This is the mean-field approximation of an important model from non-equilibrium statistical physics called the* totally asymmetric simple exclusion process *(TASEP) [28]. In the RFM, $x_i \in [0, 1]$ is the normalized occupancy at site i, where $x_i = 0$ [$x_i = 1$] means that site i is completely free [full], and η_i is the capacity of the link that connects site i to site $i + 1$. This has been used to model mRNA translation, where every site corresponds to a group of codons on the mRNA strand, $x_i(t)$ is the normalized occupancy of ribosomes at site i at time t, $c(t)$ is the initiation rate at time t, and η_i is the elongation rate from site i to site $i + 1$.*

Our original motivation for generalizing (2) *was to prove entrainment in the RFM [22]. For more results on the RFM, see [29], [30], [31], [32], [33].*

Assume that there exists $\eta_0 > 0$ such that $c(t) \geq \eta_0$ for all $t \geq t_1$. Let $\Omega := [0, p_1] \times \cdots \times [0, p_n]$ denote the state-space of (17)*. Then, as shown in Section V,* (17) *does not satisfy* (2)*, w.r.t. any norm, on Ω, yet it is SOST on Ω w.r.t. the L_1 norm.*

Considering Theorem 1 in the special case where all the sets Ω_ζ in Definition 2 are equal to Ω yields the following result.

Corollary 1 *Suppose that* (1) *is contractive on Ω w.r.t. a set of norms $|\cdot|_\zeta$, $\zeta \in (0, 1/2]$, and that condition (c) in Definition 2 holds. Then* (1) *is SOST on Ω w.r.t. $|\cdot|$.*

Corollary 1 may be useful in cases where some matrix measure of the Jacobian J of (1) turns out to be non positive on Ω, but not strictly negative, suggesting that the system is "on the verge" of satisfying (2). The next result demonstrates this for the time-invariant system

$$\dot{x} = f(x), \tag{18}$$

and the particular case of the matrix measure $\mu_1 : \mathbb{R}^{n \times n} \to \mathbb{R}$ induced by the L_1 norm. Recall that this is given by (4) with the c_js defined in (5).

Proposition 2 *Consider the Jacobian $J(\cdot) : \Omega \to \mathbb{R}^{n \times n}$ of the time-invariant system* (18)*. Suppose that Ω is compact and that the set $\{1, \ldots, n\}$ can be divided into two non-empty disjoint sets S_0 and S_- such that the following properties hold for all $x \in \Omega$:*

1) *for any $k \in S_0$, $c_k(J(x)) \leq 0$;*
2) *for any $j \in S_-$, $c_j(J(x)) < 0$;*

3) for any $i \in S_0$ there exists an index $z = z(i) \in S_-$ such that $J_{zi}(x) > 0$.
Then (18) *is SOST on Ω w.r.t. the L_1 norm.*

The proof of Proposition 2 is based on the following idea. By compactness of Ω, there exists $\delta > 0$ such that

$$c_j(J(x)) < -\delta, \quad \text{for all } j \in S_- \text{ and all } x \in \Omega. \tag{19}$$

The conditions stated in the proposition imply that there exists a diagonal matrix P such that $c_k(PJP^{-1}) < 0$ for all $k \in S_0$. Furthermore, there exists such a P with diagonal entries *arbitrarily close* to 1, so $c_j(PJP^{-1}) < -\delta/2$ for all $j \in S_-$. Thus, $\mu_1(PJP^{-1}) < 0$. Now Corollary 1 implies SOST. Note that this implies that the compactness assumption may be dropped if for example it is known that (19) holds.

Example 4 *Consider the system:*

$$\begin{aligned}\dot{x} &= -\delta x + k_1 y - k_2(e_T - y)x,\\ \dot{y} &= -k_1 y + k_2(e_T - y)x,\end{aligned} \tag{20}$$

where $\delta, k_1, k_2, e_T > 0$, and $\Omega := [0,\infty) \times [0, e_T]$. This is a basic model for a transcriptional module that is ubiquitous in both biology and synthetic biology (see, e.g., [34], [3]). Here $x(t)$ is the concentration at time t of a transcriptional factor X that regulates a downstream transcriptional module by binding to a promoter with concentration $e(t)$ yielding a protein-promoter complex Y with concentration $y(t)$. The binding reaction is reversible with binding and dissociation rates k_2 and k_1, respectively. The linear degradation rate of X is δ, and as the promoter is not subject to decay, its total concentration, e_T, is conserved, so $e(t) = e_T - y(t)$. The Jacobian of (20) *is $J = \begin{bmatrix} -\delta - k_2(e_T - y) & k_1 + k_2 x \\ k_2(e_T - y) & -k_1 - k_2 x \end{bmatrix}$, and all the properties in Prop. 2 hold with $S_- = \{1\}$ and $S_0 = \{2\}$. Indeed, $J_{12} = k_1 + k_2 x > k_1 > 0$ for all $\begin{bmatrix} x & y \end{bmatrix}^T \in \Omega$. Thus,* (20) *is SOST on Ω w.r.t. the L_1 norm. Note that Ref. [3] showed that* (20) *is contractive w.r.t. a certain* weighted *L_1 norm. Here we showed SOST w.r.t. the (unweighted) L_1 norm.*

Example 5 *A more general example studied in [3] is where the transcription factor regulates several independent downstream transcriptional modules. This leads to the following model:*

$$\begin{aligned}\dot{x} &= -\delta x + k_{11}y_1 - k_{21}(e_{T,1} - y_1)x + k_{12}y_2 - k_{22}(e_{T,2} - y_2)x + \dots + k_{1n}y_n - k_{2n}(e_{T,n} - y_n)x,\\ \dot{y}_1 &= -k_{11}y_1 + k_{21}(e_{T,1} - y_1)x,\\ &\vdots\\ \dot{y}_n &= -k_{1n}y_n + k_{2n}(e_{T,n} - y_n)x,\end{aligned} \tag{21}$$

where n is the number of regulated modules. The state-space is $\Omega = [0,\infty) \times [0, e_{T,1}] \times \dots \times [0, e_{T,n}]$. The Jacobian of (21) *is*

$$J = \begin{bmatrix} -\delta - \sum_{i=1}^n k_{2i}(e_{T,i} - y_i) & k_{11} + k_{21}x & k_{12} + k_{22}x & \dots & k_{1n-1} + k_{2n-1}x & k_{1n} + k_{2n}x \\ k_{21}(e_{T,1} - y_1) & -k_{11} - k_{21}x & 0 & \dots & 0 & 0 \\ k_{22}(e_{T,2} - y_2) & 0 & -k_{12} - k_{22}x & 0 & \dots & 0 \\ & \vdots & & & & \\ k_{2n}(e_{T,n} - y_n) & 0 & 0 & \dots & 0 & -k_{1n} - k_{2n}x \end{bmatrix},$$

and all the properties in Prop. 2 hold with $S_- = \{1\}$ and $S_0 = \{2, 3, \dots, n\}$. Thus, this system is SOST on Ω w.r.t. the L_1 norm.

Arguing as in the proof of Proposition 2 for the matrix measure μ_∞ induced by the L_∞ norm (see (7)) yields the following result.

Proposition 3 *Consider the Jacobian $J(\cdot) : \Omega \to \mathbb{R}^{n\times n}$ of the time-invariant system* (18). *Suppose that Ω is compact and that the set $\{1, \dots, n\}$ can be divided into two non-empty disjoint sets S_0 and S_- such that the following properties hold for all $x \in \Omega$:*

1) $d_j(J(x)) \le 0$ *for all* $j \in S_0$;
2) $d_k(J(x)) < 0$ *for all* $k \in S_-$;
3) *for any $j \in S_0$ there exists an index $z = z(j) \in S_-$ such that $J_{jz}(x) \ne 0$.*

Then (18) *is SOST on Ω w.r.t. the L_∞ norm.*

B. contractive after a small overshoot

A natural question is under what conditions SO and SOST are equivalent. To address this issue, we introduce the following definition.

Definition 3 *We say that* (1) *is* weakly expansive *(WE) if for each $\delta > 0$ there exists $\tau_0 > 0$ such that for all $a, b \in \Omega$ and all $t_0 \ge 0$*

$$|x(t, t_0, a) - x(t, t_0, b)| \le (1 + \delta)|a - b|, \quad \textit{for all } t \in [t_0, t_0 + \tau_0]. \tag{22}$$

Proposition 4 *Suppose that* (1) *is WE. Then* (1) *is SOST if and only if it is SO.*

Remark 1 *Suppose that f in* (1) *is Lipschitz globally in Ω uniformly in t, i.e. there exists $L > 0$ such that*

$$|f(t, x) - f(t, y)| \le L|x - y|, \quad \textit{for all } x, y \in \Omega,\ t \ge 0.$$

Then by Gronwall's Lemma (see, e.g. [35, Appendix C])

$$|x(t, t_0, a) - x(t, t_0, b)| \le \exp\left(L(t - t_0)\right)|a - b|,$$

for all $t \ge t_0 \ge 0$, and this implies that (22) *holds for $\tau_0 := \frac{1}{L}\ln(1+\delta) > 0$. In particular, if Ω is compact and f is periodic in t then WE holds under rather weak continuity arguments on f.*

C. contractive after a short transient

For *time-invariant* systems whose state evolves on a convex and compact set it is possible to give a simple sufficient condition for ST. Let $\mathrm{Int}(S)$ [∂S] denote the interior [boundary] of a set S. We require the following definitions.

Definition 4 *We say that* (1) *is* non expansive *(NE) w.r.t. a norm $|\cdot|$ if for all $a, b \in \Omega$ and all $s_2 > s_1 \ge 0$*

$$|x(s_2, s_1, a) - x(s_2, s_1, b)| \le |a - b|. \tag{23}$$

We say that (1) *is* weakly contractive *(WC) if* (23) *holds with $\le$ replaced by $<$.*

Definition 5 *The time-invariant system* (18) *with the state x evolving on a compact and convex set $\Omega \subset \mathbb{R}^n$, is said to be* interior contractive *(IC) w.r.t. a norm $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ if the following properties hold:*

(a) for every $x_0 \in \partial\Omega$,

$$x(t, x_0) \notin \partial\Omega, \quad \textit{for all } t > 0; \tag{24}$$

(b) for every $x \in \mathrm{Int}(\Omega)$,

$$\mu(J(x)) < 0, \tag{25}$$

where $\mu : \mathbb{R}^{n\times n} \to \mathbb{R}$ is the matrix measure induced by $|\cdot|$.

In other words, the matrix measure is negative in the interior of Ω, and the boundary of Ω is "repelling". Note that these conditions do not necessarily imply that the system satisfies (2) on Ω, as it is possible

that $\mu(J(x)) = 0$ for some $x \in \partial\Omega$. Yet, (25) does imply that (18) is NE on Ω. We can now state the main result in this subsection.

Theorem 2 *If the system* (18) *is IC w.r.t. a norm* $|\cdot|$ *then it is ST w.r.t.* $|\cdot|$.

The proof of this result is based on showing that IC implies that for each $\tau > 0$ there exists $d = d(\tau) > 0$ such that

$$\operatorname{dist}(x(t, x_0), \partial\Omega) \geq d, \quad \text{for all } x_0 \in \Omega \text{ and all } t \geq \tau,$$

and then using this to conclude that for any $t \geq \tau$ all the trajectories of the system are contained in a convex and compact set $D \subset \operatorname{Int}(\Omega)$. In this set the system is contractive with rate $c := \max_{x\in D} \mu(J(x)) < 0$. The next example, that is a variation of a system studied in [3], demonstrates this reasoning.

Example 6 *Consider a transcriptional factor X that regulates a downstream transcriptional module by irreversibly binding, at a rate $k_2 > 0$, to a promoter E yielding a protein-promoter complex Y. The promoter is not subject to decay, so its total concentration, denoted by $e_T > 0$, is conserved. Assume also that X is obtained from an inactive form X_0, for example through a phosphorylation reaction that is catalyzed by a kinase with abundance $u(t)$ satisfying $u(t) \geq u_0 > 0$ for all $t \geq 0$. The sum of the concentrations of X_0, X, and Y is constant, denoted by z_T, with $z_T > e_T$. Letting $x_1(t), x_2(t)$ denote the concentrations of X, Y at time t yields the model*

$$\begin{aligned} \dot{x}_1 =& (z_T - x_1 - x_2)u - \delta x_1 - k_2(e_T - x_2)x_1, \\ \dot{x}_2 =& k_2(e_T - x_2)x_1, \end{aligned} \tag{26}$$

with the state evolving on $\Omega := [0, z_T] \times [0, e_T]$. Here $\delta \geq 0$ is the dephosphorylation rate $X \to X_0$. Let $P := \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, and consider the matrix measure $\mu_{\infty,P}$. A calculation yields

$$\begin{aligned} \tilde{J} &:= PJP^{-1} \\ &= \begin{bmatrix} -u - \delta & \delta \\ k_2(e_T - x_2) & k_2(x_2 - x_1 - e_T) \end{bmatrix}, \end{aligned}$$

so $d_1(\tilde{J}) = -u - \delta + |\delta| \leq -u_0 < 0$, and

$$\begin{aligned} d_2(\tilde{J}) &= k_2(x_2 - x_1 - e_T) + |k_2(e_T - x_2)| \\ &= -k_2 x_1. \end{aligned}$$

Letting $S := \{0\} \times [0, e_T]$, we conclude that $\mu_{\infty,P}(x) < 0$ for all $x \in (\Omega \setminus S)$. For any $x \in S$, $\dot{x}_1 = (z_T - x_2)u \geq (z_T - e_T)u_0 > 0$, and arguing as in the proof of Theorem 2 (see Section V), we conclude that for any $\tau > 0$ there exists $d = d(\tau) > 0$ such that

$$x_1(t, a) \geq d, \quad \textit{for all } a \in \Omega \textit{ and all } t \geq \tau.$$

In other words, after time τ all the trajectories are contained in the closed and convex set $D = D(\tau) := [d, z_T] \times [0, e_T]$. Letting $c := c(\tau) = \max_{x\in D} \mu_{\infty,P}(J(x))$ yields $c < 0$ and

$$|x(t + \tau, a) - x(t + \tau, b)|_{\infty,P} \leq \exp(ct)|a - b|_{\infty,P}, \quad \textit{for all } a, b \in \Omega \textit{ and all } t > 0,$$

so (26) *is ST w.r.t.* $|\cdot|_{\infty,P}$.

As noted above, the introduction of GCSs is motivated by the idea that contraction is used to prove asymptotic results, so allowing initial transients should increase the class of systems that can be analyzed while still allowing to prove asymptotic results. The next result demonstrates this.

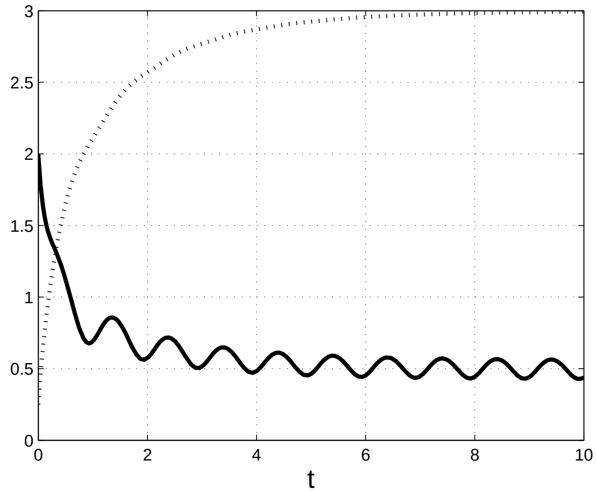

Fig. 1. Solution $x_1(t)$ (solid line) and $x_2(t)$ (dashed line) of the system in Example 7 as a function of t.

Corollary 2 *If* (18) *is IC with respect to some norm then it admits a unique equilibrium point* $e \in \mathrm{Int}(\Omega)$*, and* $\lim_{t\to\infty} x(t,a) = e$ *for all* $a \in \Omega$*.*

Remark 2 *The proof of Corollary 2, given in the Appendix, is based on Theorem 2. Consider the* variational system *(see, e.g., [12]) associated with* (18)*:*

$$\begin{aligned} \dot{x} &= f(x), \\ \dot{\delta x} &= J(x)\delta x. \end{aligned} \tag{27}$$

An alternative proof of Corollary 2 is possible, using the Lyapunov-Finsler function $V(x, \delta x) := |\delta x|$*, where* $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ *is the vector norm corresponding to the matrix measure* μ *in* (25)*, and the LaSalle invariance principle described in [12].*

Since IC implies ST and this implies SOST, it follows from Proposition 1 that IC implies entrainment to T-periodic vector fields.[1] The next example demonstrates this.

Example 7 *Consider again the system in Example 6, and assume that the kinase abundance* $u(t)$ *is a strictly positive and periodic function of time with period* T*. Since we already showed that this system is ST, it admits a unique periodic solution* γ*, of period* T*, and any trajectory of the system converges to* γ*. Figure 1 depicts the solution of* (26) *for* $\delta = 2$*,* $k_2 = 1$*,* $z_T = 4$*,* $e_T = 3$*,* $u(t) = 2 + \sin(2\pi t)$*, and initial condition* $x_1(0) = 2, x_2(0) = 1/4$*. It may be seen that both state-variables converge to a periodic solution with period* $T = 1$*. (In particular,* x_2 *converges to the constant function* $x_2(t) \equiv e_T$ *that is of course periodic with period* T*.)*

Contraction can be characterized using a Lyapunov-Finsler function [12]. The next result describes a similar characterization for ST. For simplicity, we state this for the time-invariant system (18).

Proposition 5 *The following two conditions are equivalent.*
(a) The time-invariant system (18) *is ST w.r.t. a norm* $|\cdot|$*.*

[1]Note that the proof that IC implies ST used a result for time-invariant systems, but an analogous argument holds for the time-varying case as well.

(b) For any $\tau > 0$ there exists $\ell = \ell(\tau) > 0$ such that for any $a, b \in \Omega$ and any c on the line connecting a and b the solution of (27) *with $x(0) = c$ and $\delta x(0) = b - a$ satisfies*

$$|\delta x(t+\tau)| \le \exp(-\ell t)|\delta x(0)|, \quad \textit{for all } t \ge 0. \tag{28}$$

Note that (28) implies that the function $V(x, \delta x) := |\delta x|$ is a *generalized* Lyapunov-Finsler function in the following sense. For any $\tau > 0$ there exists $\ell = \ell(\tau) > 0$ such that along solutions of the variational system:

$$V\left(x(t+\tau, x(0)), \delta x(t+\tau, \delta x(0), x(0))\right) \le \exp(-\ell t) V(x(0), \delta x(0)), \quad \text{for all } t \ge 0.$$

In the next section, we describe several more related notions and explore the relations between them.

IV. Additional Notions and Relations

It is straightforward to show that each of the three generalizations of contraction in Definition 1 implies that (1) is NE. One may perhaps expect that any of the three generalizations of contraction in Definition 1 also implies WC. Indeed, ST does imply WC, because

$$|x(s_2, s_1, a) - x(s_2, s_1, b)| \le \exp\left(-\ell(s_2 - s_1)/2\right)|a - b| < |a - b|,$$

for all $0 \le s_1 < s_2$ if ST holds (simply apply the definition with $t_1 = s_1$, $\tau = (s_2 - s_1)/2 > 0$, and $t_2 = s_1 + \tau$ in (10)). However, the next example shows that SO does not imply WC.

Example 8 *Consider the scalar system*

$$\dot{x} = \begin{cases} -2x, & 0 \le |x| < 1/2, \\ -\frac{x}{|x|}, & \frac{1}{2} \le |x| \le 1, \end{cases} \tag{29}$$

with x evolving on $\Omega := [-1, 1]$. Clearly, this system is not WC. However, it is not difficult to show that it satisfies the definition of SO with $\ell = \ell(\varepsilon) := \min\{\ln(1+\varepsilon), 1\}$.

The next result presents two conditions that are equivalent to SOST.

Lemma 1 *The following conditions are equivalent.*

1) *System* (1) *is SOST on Ω w.r.t. some vector norm $|\cdot|_v : \mathbb{R}^n \to \mathbb{R}_+$.*
2) *For each $\tau > 0$ there exists $\ell = \ell(\tau) > 0$ such that*

$$|x(t_2+\tau, t_1, a) - x(t_2+\tau, t_1, b)|_v \le (1+\tau)\exp(-(t_2 - t_1)\ell)|a - b|_v, \tag{30}$$

for all $t_2 \ge t_1 \ge 0$ and all $a, b \in \Omega$.

3) *For each $\varepsilon > 0$ and each $\tau > 0$ there exists $\ell_1 = \ell_1(\tau, \varepsilon) > 0$ such that*

$$|x(t, t_1, a) - x(t, t_1, b)|_v \le (1+\varepsilon)\exp(-(t - t_1)\ell_1)|a - b|_v, \tag{31}$$

for all $t \ge t_1 + \tau \ge \tau$ and all $a, b \in \Omega$.

Fig. 2 summarizes the relations between the various contraction notions.

V. Proofs

Proof of Theorem 1. Fix arbitrary $\varepsilon > 0$ and $t_1 \ge 0$. The function $\zeta = \zeta(\tau) \in (0, 1/2]$ is as in the statement of the Theorem. For each $\tau > 0$, let $c_\zeta > 0$ be a contraction constant on Ω_ζ, where we write $\zeta = \zeta(\tau)$ here and in what follows. Pick $a, b \in \Omega$ and $\tau > 0$. By (13), $x(t, t_1, a), x(t, t_1, b) \in \Omega_\zeta$ for all $t \ge t_1 + \tau$, so

$$|x(t, t_1, a) - x(t, t_1, b)|_\zeta \le \exp(-c_\zeta(t - t_1 - \tau))|x(t_1+\tau, t_1, a) - x(t_1+\tau, t_1, b)|_\zeta, \quad \text{for all } t \ge t_1 + \tau. \tag{32}$$

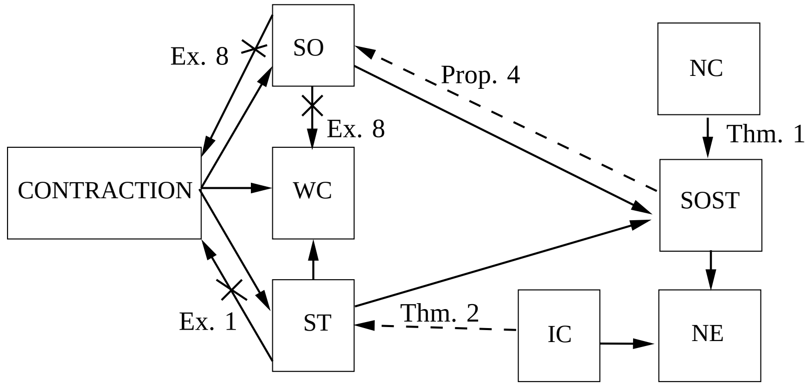

Fig. 2. Relations between various contraction notions. An arrow denotes implication; a crossed out arrow denotes that the implication is in general false; and a dashed arrow denotes an implication that holds under an additional condition. Some of the relations are immediate. Others follow from the results marked near the arrows.

In particular,

$$|x(t,t_1,a)-x(t,t_1,b)|_\zeta < |x(t_1+\tau,t_1,a)-x(t_1+\tau,t_1,b)|_\zeta, \quad \text{for all } t>t_1+\tau. \tag{33}$$

From the convergence property of norms in the Theorem statement, there exist $v_\zeta, w_\zeta > 0$ such that

$$|y| \le v_\zeta |y|_\zeta \le w_\zeta v_\zeta |y|, \quad \text{for all } y \in \Omega, \tag{34}$$

and $v_\zeta \to 1$, $w_\zeta \to 1$ as $\tau \to 0$. Combining this with (33) yields

$$|x(t,t_1,a)-x(t,t_1,b)| < v_\zeta w_\zeta |x(t_1+\tau,t_1,a)-x(t_1+\tau,t_1,b)|, \quad \text{for all } t>t_1+\tau.$$

Note that taking $\tau \to 0$ yields

$$|x(t,t_1,a)-x(t,t_1,b)| \le |a-b|, \quad \text{for all } t>t_1. \tag{35}$$

Now for $t \ge t_1+\tau$ let $p := t-t_1-\tau$. Then

$$\begin{aligned} |x(t,t_1,a)-x(t,t_1,b)| &\le v_\zeta |x(t,t_1,a)-x(t,t_1,b)|_\zeta \\ &\le v_\zeta \exp(-c_\zeta p)|x(t_1+\tau,t_1,a)-x(t_1+\tau,t_1,b)|_\zeta \\ &\le v_\zeta w_\zeta \exp(-c_\zeta p)|x(t_1+\tau,t_1,a)-x(t_1+\tau,t_1,b)| \\ &\le v_\zeta w_\zeta \exp(-c_\zeta p)|a-b|, \end{aligned}$$

where the last inequality follows from (35). Since $v_\zeta \to 1$, $w_\zeta \to 1$ as $\tau \to 0$, $v_\zeta w_\zeta \le 1+\varepsilon$ for $\tau > 0$ small enough. Summarizing, there exists $\tau_m = \tau_m(\varepsilon) > 0$ such that for all $\tau \in [0,\tau_m]$

$$|x(t+\tau,t_1,a)-x(t+\tau,t_1,b)| \le (1+\varepsilon)\exp(-c_\zeta(t-t_1))|a-b|, \tag{36}$$

for all $a,b \in \Omega$ and all $t \ge t_1$. Now pick $\tau > \tau_m$. For any $t \ge t_1$, let $s := t+\tau-\tau_m$. Then

$$\begin{aligned} |x(t+\tau,t_1,a)-x(t+\tau,t_1,b)| &= |x(s+\tau_m,t_1,a)-x(s+\tau_m,t_1,b)| \\ &\le (1+\varepsilon)\exp(-c_\zeta(s-t_1))|a-b| \\ &\le (1+\varepsilon)\exp(-c_\zeta(t-t_1))|a-b|, \end{aligned}$$

and this completes the proof. ■

Analysis of the system from Example 2. The Jacobian of (14) is

$$J(x) = \begin{bmatrix} -\alpha_1 & 0 & 0 & \dots & 0 & g'(x_n) \\ 1 & -\alpha_2 & 0 & \dots & 0 & 0 \\ 0 & 1 & -\alpha_3 & \dots & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \dots & 1 & -\alpha_n \end{bmatrix}, \tag{37}$$

so

$$D_\varepsilon J(x) D_\varepsilon^{-1} = \begin{bmatrix} -\alpha_1 & 0 & 0 & \dots & 0 & \frac{g'(x_n)}{\prod_{i=1}^{n-1}(\alpha_i-\varepsilon)} \\ \alpha_1-\varepsilon & -\alpha_2 & 0 & \dots & 0 & 0 \\ 0 & \alpha_2-\varepsilon & 0 & \dots & 0 & 0 \\ & & \vdots & & & \\ 0 & 0 & 0 & \dots & \alpha_{n-1}-\varepsilon & -\alpha_n \end{bmatrix}.$$

Thus,

$$\mu_{1,D_\varepsilon}(J(x)) = \max\{-\varepsilon, \frac{g'(x_n) - \alpha_n \prod_{i=1}^{n-1}(\alpha_i-\varepsilon)}{\prod_{i=1}^{n-1}(\alpha_i-\varepsilon)}\}. \tag{38}$$

Suppose that $k-1 < \alpha k^2$. Then for all $x \in \mathbb{R}^n_+$,

$$g'(x_n) = \frac{k-1}{(k+x_n)^2} \le \frac{k-1}{k^2} < \alpha.$$

Combining this with (38) implies that there exists a sufficiently small $\varepsilon > 0$ such that $\mu_{1,D_\varepsilon}(J(x)) < -\varepsilon/2$ for all $x \in \mathbb{R}^n_+$, so the system is contractive on $\mathbb{R}^n_+$ w.r.t. $|\cdot|_{1,D_\varepsilon}$.

Now assume that $k-1 = \alpha k^2$. By (37),

$$\det(J(x)) = (-1)^n(\alpha - g'(x_n)),$$

so for every $x \in \mathbb{R}^n_+$ with $x_n = 0$, we have $\det(J(x)) = (-1)^n(\alpha - g'(0)) = 0$. This implies that the system does not satisfy (2), w.r.t. any norm, on $\mathbb{R}^n_+$.

We now use Theorem 1 to prove that (14) is SOST. Since $g'(u) = \frac{k-1}{(k+u)^2}$ and $k > 1$,

$$g(x_n) \ge g(0) = 1/k, \quad \text{for all } x \in \mathbb{R}^n_+.$$

For $\zeta \in (0, 1/2]$, let

$$\Omega_\zeta := \{x \in \mathbb{R}^n_+ : x \ge \zeta\}.$$

It is straightforward to verify that (14) satisfies condition (BR) in [22, Lemma 1], and this implies that for every $\tau > 0$ there exists $\varepsilon(\tau) > 0$ such that $x(t) \in \Omega_\varepsilon$ for all $t \ge \tau$. Then

$$g'(x_n) = \frac{k-1}{(k+x_n)^2} \le \frac{k-1}{(k+\varepsilon)^2} < \frac{k-1}{k^2} = \alpha.$$

We already showed that this implies that there exists a $\zeta > 0$ and a norm $|\cdot|_{1,D_\zeta}$ such that (14) is contractive on Ω_ε w.r.t. this norm. Summarizing, all the conditions in Theorem 1 hold, and we conclude that (14) is SOST on $\mathbb{R}^n_+$ w.r.t. $|\cdot|_{1,D_0}$. ■

Analysis of the system in Example 3. For $a \in \Omega$, let $x(t, t_1, a)$ denote the solution of (17) at time $t \ge t_1$ for the initial condition $x(t_1) = a$. Pick $\tau > 0$. Eq. (17) satisfies condition (BR) in [22, Lemma 1], and this implies that there exists $\varepsilon = \varepsilon(\tau) > 0$ such that for all $a \in \Omega$, all $i = 1, \dots n$, and all $t \ge t_1 + \tau$

$$x_i(t, t_1, a) \ge \varepsilon.$$

Furthermore, if we define $y_i(t) := p_{n-i+1} - x_{n-i+1}(t)$, $i = 1, \dots, n$, then the y system also satisfies condition (BR) in [22, Lemma 1], and this implies that there exists $\varepsilon_1 = \varepsilon_1(\tau) > 0$ such that for all $a \in \Omega$, all $i = 1, \dots n$, and all $t \geq t_1 + \tau$

$$y_i(t, t_1, a) \geq \varepsilon_1.$$

We conclude that after an arbitrarily short time $\tau > 0$ every state-variable $x_i(t)$, $t \geq \tau + t_1$, is separated from 0 and from p_i. This means the following. For $\zeta \in [0, 1/2]$, let

$$\Omega_\zeta := \{x \in \Omega : \ \zeta p_i \leq x_i \leq (1-\zeta) p_i, \ i = 1, \dots, n\}.$$

Note that $\Omega_0 = \Omega$, and that Ω_ζ is a strict subcube of Ω for all $\zeta \in (0, 1/2]$. Then for any $t_1 \geq 0$, and any $\tau > 0$ there exists $\zeta = \zeta(\tau) \in (0, 1/2)$, with $\zeta(\tau) \to 0$ as $\tau \to 0$, such that

$$x(t, t_1, a) \in \Omega_\zeta, \quad \text{for all } t \geq t_1 + \tau \text{ and all } a \in \Omega. \tag{39}$$

The Jacobian of (17) satisfies $J(t, x) = L(x) - \operatorname{diag}(c(t), 0, \dots, 0, \eta_n)$, where

$$L(x) = \begin{bmatrix} -\eta_1(p_2 - x_2) & \eta_1 x_1 & 0 & 0 \\ \eta_1(p_2 - x_2) & -\eta_1 x_1 - \eta_2(p_3 - x_3) & \dots & 0 \\ 0 & \eta_2(p_3 - x_3) & \dots & 0 \\ & & \ddots & \\ 0 & \dots & -\eta_{n-2} x_{n-2} - \eta_{n-1}(p_n - x_n) & \eta_{n-1} x_{n-1} \\ 0 & \dots & \eta_{n-1}(p_n - x_n) & -\eta_{n-1} x_{n-1} \end{bmatrix}.$$

Note that $L(x)$ is Metzler, tridiagonal, and has zero sum columns for all $x \in \Omega$. Note also that for any $x \in \Omega_\zeta$ every entry L_{ij} on the sub- and super-diagonal of L satisfies $\zeta s_1 \leq L_{ij} \leq (1-\zeta) s_2$, with $s_2 := \max_i \{\eta_i p_i\} > s_1 := \min_i \{\eta_i p_i\} > 0$.

Note also that there exist $x \in \partial\Omega$ such that $J(x)$ is singular (e.g., when $x_1 = 0$ and $x_3 = p_3$ the second column of J is all zeros), and this implies that the system does not satisfy (2) on Ω w.r.t. any norm.

By [22, Theorem 4], for any $\zeta \in (0, 1/2]$ there exists $\varepsilon = \varepsilon(\zeta) > 0$, and a diagonal matrix $D = \operatorname{diag}(1, q_1, q_1 q_2, \dots, q_1 q_2 \dots q_{n-1})$, with $q_i = q_i(\varepsilon) > 0$, such that (17) is contractive on Ω_ζ w.r.t. the the scaled L_1 norm defined by $|z|_{1,D} = |Dz|_1$. Furthermore, we can choose ε such that $\varepsilon(\zeta) \to 0$ as $\zeta \to 0$, and $D(\varepsilon) \to I$ as $\varepsilon \to 0$. Summarizing, all the conditions in Definition 2 hold, so (17) is NC on Ω and applying Theorem 1 concludes the analysis. ∎

Proof of Proposition 2. Without loss of generality, assume that $S_0 = \{1, \dots, k\}$, with $1 \leq k < n-1$, so that $S_- = \{k+1, \dots, n\}$. Fix $\varepsilon \in (0, 1)$. Let $D = \operatorname{diag}(d_1, \dots, d_n)$ with the d_is defined as follows. For every $i \in S_0$, $d_i = 1$ and $d_{z(i)} = 1 - \varepsilon$. All the other d_is are one. Let $\tilde{J} := D J D^{-1}$. Then $\tilde{J}_{ij} = \frac{d_i}{d_j} J_{ij}$. We now calculate $\mu_1(\tilde{J})$. Fix $j \in S_0$. Then $d_j = 1$, so

$$\begin{aligned} c_j(\tilde{J}) &= \tilde{J}_{jj} + \sum_{\substack{1 \leq i \leq n \\ i \neq j}} |\tilde{J}_{ij}| \\ &= J_{jj} + \sum_{\substack{i \in S_0 \\ i \neq j}} d_i |J_{ij}| + \sum_{\substack{k \in S_- \\ k \neq j}} d_k |J_{kj}| \\ &= J_{jj} + \sum_{\substack{i \in S_0 \\ i \neq j}} |J_{ij}| + \sum_{\substack{k \in S_- \\ k \neq j}} d_k |J_{kj}| \\ &< c_j(J), \end{aligned}$$

where the inequality follows from the fact that $d_k \leq 1$ for all k, and for the specific value $k = z(j) \in S_-$ we have $d_k = 1 - \varepsilon$ and $|J_{kj}| > 0$. We conclude that for every $j \in S_0$, $c_j(\tilde{J}) < c_j(J) = 0$. It follows

from property 2) in the statement of Proposition 2 and the compactness of Ω that there exists $\delta > 0$ such that $c_j(J(x)) < -\delta$ for all $j \in S_-$ and all $x \in \Omega$, so for $\varepsilon > 0$ sufficiently small we have $c_j(\tilde{J}(x)) < -\delta/2$ for all $j \in S_-$ and all $x \in \Omega$. We conclude that for all $\varepsilon > 0$ sufficiently small, $\mu_1(DJD^{-1}) = \max_j c_j(\tilde{J}) < 0$, i.e. the system is contractive w.r.t. $|\cdot|_{1,D}$. Clearly, $|\cdot|_{1,D} \to |\cdot|_1$ as $\varepsilon \to 0$, and applying Corollary 1 completes the proof. ■

Proof of Proposition 4. Suppose that (1) is SOST w.r.t. some norm $|\cdot|_v$. Pick $\varepsilon > 0$. Since the system is WE, there exists $\tau_0 = \tau_0(\varepsilon) > 0$ such that

$$|x(t,t_0,a) - x(t,t_0,b)|_v \le (1+\varepsilon/2)|a-b|_v,$$

for all $t \in [t_0, t_0+\tau_0]$. Letting $\ell_2 := \frac{1}{\tau_0}\ln(\frac{1+\varepsilon}{1+(\varepsilon/2)})$ yields

$$|x(t,t_0,a) - x(t,t_0,b)|_v \le (1+\varepsilon)\exp(-(t-t_0)\ell_2)|a-b|_v, \tag{40}$$

for all $t \in [t_0, t_0+\tau_0]$. By item 3 in Lemma 1 there exists $\ell_1 = \ell_1(\tau_0, \varepsilon) > 0$ such that

$$|x(t,t_0,a) - x(t,t_0,b)|_v \le (1+\varepsilon)\exp(-(t-t_0)\ell_1)|a-b|_v,$$

for all $t \ge t_0 + \tau_0$. Combining this with (40) yields

$$|x(t,t_0,a) - x(t,t_0,b)|_v \le (1+\varepsilon)\exp(-(t-t_0)\ell)|a-b|_v,$$

for all $t \ge t_0$, where $\ell := \min\{\ell_1, \ell_2\} > 0$. This proves SO. ■

Proof of Theorem 2. We require the following result.

Lemma 2 *If system* (18) *is IC then for each $\tau > 0$ there exists $d = d(\tau) > 0$ such that*

$$\operatorname{dist}(x(t,x_0), \partial\Omega) \ge d, \quad \textit{for all } x_0 \in \Omega \textit{ and all } t \ge \tau.$$

Proof of Lemma 2. Pick $\tau > 0$ and $x_0 \in \Omega$. Since Ω is an invariant set, $\operatorname{Int}(\Omega)$ is also an invariant set (see, e.g., [36, Lemma III.6]), so (24) implies that $x(t,x_0) \notin \partial\Omega$ for all $t > 0$. Since $\partial\Omega$ is compact, $e_{x_0} := \operatorname{dist}(x(\tau,x_0), \partial\Omega) > 0$. Thus, there exists a neighborhood U_{x_0} of x_0, such that $\operatorname{dist}(x(\tau,y), \partial\Omega) \ge e_{x_0}/2$ for all $y \in U_{x_0}$. Cover Ω by such U_{x_0} sets. By compactness of Ω, we can pick a finite subcover. Pick smallest e in this subcover, and denote this by d. Then $d > 0$ and we have that $\operatorname{dist}(x(\tau,x_0), \partial\Omega) \ge d$ for all $x_0 \in \Omega$. Now, pick $t \ge \tau$. Let $x_1 := x(t-\tau, x_0)$. Then

$$\begin{aligned}\operatorname{dist}(x(t,x_0), \partial\Omega) &= \operatorname{dist}(x(\tau,x_1), \partial\Omega) \\ &\ge d,\end{aligned}$$

and this completes the proof of Lemma 2. ■

We can now prove Theorem 2. We recall some definitions from the theory of convex sets. Let $B(x,r)$ denote the closed ball of radius r around x (in the Euclidean norm). Let K be a compact and convex set with $0 \in \operatorname{Int}(K)$. Let $s(K)$ denote the *inradius* of k, i.e. the radius of the largest ball contained in K. For $\lambda \in [0, s(K)]$ the *inner parallel set of K at distance λ* is

$$K_{-\lambda} := \{x \in K : B(x,\lambda) \subseteq K\}.$$

Note that $K_{-\lambda}$ is a compact and convex set; in fact, $K_{-\lambda}$ is the intersection of all the translated support hyperplanes of K, with each hyperplane translated “inwards” through a distance λ (see [37, Section 17]). Assume, without loss of generality, that $0 \in \operatorname{Int}(\Omega)$. Pick $\tau > 0$. Let $M = M(\tau) := \{x(t,x_0) : t \ge \tau,\ x_0 \in \Omega\}$. By Lemma 2, $M \subset \Omega$ and $\operatorname{dist}(y, \partial\Omega) \ge d > 0$ for all $y \in M$. Let $\lambda = \lambda(\tau) := \frac{1}{2}\min\{d, s(\Omega)\}$. Then $\lambda > 0$. Pick $z \in M$. We claim that $B(z,\lambda) \subseteq \Omega$. To show this, assume that there exists $v \in B(z,\lambda)$

such that $v \notin \Omega$. Then there is a point q on the line connecting v and z such that $q \in \partial\Omega$. Therefore,

$$\begin{aligned}\operatorname{dist}(z, \partial\Omega) &\leq |z-q| \\ &\leq |z-v| \\ &\leq \lambda \\ &\leq d/2,\end{aligned}$$

and this is a contradiction as $z \in M$. We conclude that $M \subseteq K_{-\lambda}$. Let $c = c(\tau) := \max_{x\in K_{-\lambda}} \mu(J(x))$. Then (25) implies that $c < 0$. Thus, the system is contractive on $K_{-\lambda}$, and for all $a, b \in \Omega$ and all $t \geq 0$

$$|x(t+\tau, a) - x(t+\tau, b)| \leq \exp(ct)|a-b|,$$

where $|\cdot|$ is the vector norm corresponding to the matrix measure μ. This establishes ST, and thus completes the proof of Theorem 2. ■

Proof of Corollary 2. Since Ω is convex, compact, and invariant, it includes an equilibrium point e of (18). Clearly, $e \in \operatorname{Int}(\Omega)$. By Theorem 2, the system is ST. Pick $a \in \Omega$ and $\tau > 0$, and let $\ell = \ell(\tau) > 0$. Applying (10) with $b = e$ yields

$$|x(t+\tau, a) - e| \leq \exp(-\ell t)|a-e|,$$

for all $t \geq 0$. Taking $t \to \infty$ completes the proof. ■

Remark 3 *Another possible proof of Corollary 2 is based on defining $V : \Omega \to \mathbb{R}_+$ by $V(x) := |x-e|$. Then for any $a \in \Omega$, $V(x(t,a))$ is nondecreasing, and the LaSalle invariance principle tells us that $x(t,a)$ converges to an invariant subset of the set $\{y \in \Omega : |y-e| = r\}$, for some $r \geq 0$. If $r = 0$ then we are done. Otherwise, pick y in the omega limit set of the trajectory. Then $y \notin \partial\Omega$, so* (25) *implies that V is strictly decreasing. This contradiction completes the proof.*

Proof of Proposition 5. Pick $a, b \in \Omega$. Let $\gamma : [0,1] \to \Omega$ be the line $\gamma(r) := (1-r)a + rb$. Note that since Ω is convex, $\gamma(r) \in \Omega$ for all $r \in [0,1]$. Let

$$w(t,r) := \frac{d}{dr} x(t, \gamma(r)).$$

This measures the sensitivity of the solution at time t to a change in the initial condition along the line γ. Note that $w(0,r) = \frac{d}{dr}\gamma(r) = b - a$, and

$$\dot{w}(t,r) = J(x(t,\gamma(r)))w(t,r).$$

Comparing this to (27) implies that $w(t,r)$ is equal to the second component, $\delta x(t)$, of the solution of the variational system (27) with initial condition

$$\begin{aligned} x(0) &= (1-r)a + rb, \\ \delta x(0) &= b - a. \end{aligned} \tag{41}$$

Suppose that the time-invariant system (18) is ST. Pick $\tau > 0$. Let $\ell = \ell(\tau) > 0$. Then for any $r \in [0,1)$ and any $\varepsilon \in [0, 1-r]$,

$$|x(t+\tau, \gamma(r+\varepsilon)) - x(t+\tau, \gamma(r))| \leq \exp(-t\ell)|\gamma(r+\varepsilon) - \gamma(r)|.$$

Dividing both sides of this inequality by ε and taking $\varepsilon \downarrow 0$ implies that

$$|w(t+\tau, r)| \leq \exp(-t\ell)|b-a|, \tag{42}$$

so

$$|\delta x(t+\tau)| \leq \exp(-t\ell)|\delta x(0)|.$$

This proves the implication (a) $\to$ (b). To prove the converse implication, assume that (28) holds. Then (42) holds and thus

$$\begin{aligned}|x(t+\tau,b)-x(t+\tau,a)| &= \left|\int_0^1 \frac{d}{dr}x(t+\tau,\gamma(r))\mathrm{d}r\right| \\ &\le \int_0^1 |w(t+\tau,r)|\,\mathrm{d}r \\ &\le \int_0^1 \exp(-\ell t)|b-a|\mathrm{d}r \\ &= \exp(-\ell t)|b-a|,\end{aligned}$$

so the system is ST. ■

Proof of Lemma 1. If (1) is SOST then (30) holds for the particular case $\varepsilon=\tau$ in Definition 1. To prove the converse implication, assume that (30) holds. Pick $\hat{\tau},\hat{\varepsilon}>0$. Let

$$\tau := \min\{\hat{\tau},\hat{\varepsilon}\}, \tag{43}$$

and let $\ell=\ell(\tau)>0$. Pick $t\ge t_1\ge 0$, and let $t_2:=t+\hat{\tau}-\tau\ge t_1$. Then

$$\begin{aligned}|x(t_2+\tau,t_1,a)-x(t_2+\tau,t_1,b)|_v &\le (1+\tau)\exp(-(t_2-t_1)\ell)|a-b|_v \\ &\le (1+\hat{\varepsilon})\exp(-(t-t_1)\ell)|a-b|_v,\end{aligned}$$

where the last inequality follows from (43). Thus,

$$|x(t+\hat{\tau},t_1,a)-x(t+\hat{\tau},t_1,b)|_v \le (1+\hat{\varepsilon})\exp(-(t-t_1)\ell)|a-b|_v,$$

and recalling that $\hat{\tau},\hat{\varepsilon}>0$ were arbitrary, we conclude that Condition 2) in Lemma 1 implies SOST.

To prove that Condition 3) is equivalent to SOST, suppose that (31) holds. Then for any $t_2\ge t_1$,

$$\begin{aligned}|x(t_2+\tau,t_1,a)-x(t_2+\tau,t_1,b)|_v &\le (1+\varepsilon)\exp(-(t_2+\tau-t_1)\ell_1)|a-b|_v \\ &\le (1+\varepsilon)\exp(-(t_2-t_1)\ell_1)|a-b|_v,\end{aligned}$$

so we have SOST. Conversely, suppose that (1) is SOST. Pick any $\tau,\varepsilon>0$. Then there exists $\ell=\ell(\tau,\varepsilon/2)>0$ such that for any $t\ge t_1+\tau$

$$\begin{aligned}|x(t,t_1,a)-x(t,t_1,b)|_v &= |x(t-\tau+\tau,t_1,a)-x(t-\tau+\tau,t_1,b)|_v \\ &\le (1+\varepsilon/2)\exp(-(t-\tau-t_1)\ell)|a-b|_v.\end{aligned}$$

Thus, for any $c\in(0,1)$

$$|x(t,t_1,a)-x(t,t_1,b)|_v \le (1+\varepsilon/2)\exp(\tau c\ell)\exp(-(t-t_1)c\ell)|a-b|_v.$$

Taking $c>0$ sufficiently small such that $(1+\varepsilon/2)\exp(\tau c\ell)\le 1+\varepsilon$ implies that (31) holds for $\ell_1:=c\ell$. This completes the proof that (31) is equivalent to SOST. ■

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